ON THE CANONICAL FORM OF A NON-SINGULAR PENCIL OF HERMITIAN MATRICES

A Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

G. RICHARD TROTT

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ON THE CANONICAL FORM OF A NON-SINGULAR PENCIL OF HERMITIAN MATRICES.

G. Richard Trott.

Introduction. In this paper we shall be interested in finding a set of normal forms to one of which a non-singular pencil of Hermitian matrices can be reduced by a conjunctive transformation. The result may be used immediately for finding a necessary and sufficient condition that two non-singular pencils of Hermitian matrices be equivalent under conjunctive transformations. As has been done in previous cases we carry out the work simultaneously for a non-singular pencil of real symmetric matrices under real congruent transformations and a non-singular pencil of Hermitian matrices under conjunctive transformations.

1. In order to simplify our later arguments we first define certain notations and prove some preliminary Lemmas on matrices. If R is a matrix of order n, we may consider R as a matrix of matrices and write

$$R = (R_{ij}), \qquad (i, j = 1, 2, \cdots, t)$$

where R_{ij} is a matrix of r_i rows and r_j columns and

$$r_1 + r_2 + r_3 + \cdots + r_t = n.$$

In particular, if $R_{ij} = 0$, where $i \neq j$, we shall call the matrix a diagonal block matrix and denote it by

$$R = [R_{11}, R_{22}, \cdots, R_{tt}].$$

If S is another square matrix of order n, and S is written as a matrix of matrices

$$S = (S_{ij}),$$

where S_{ij} is a matrix of r_i rows and r_j columns, we shall say that S and R are similarly partitioned.

We shall represent by $(f_0, f_1, \cdots, f_{j-1})_i^j$ the matrix of i rows and j columns, where f_0 is the element in each place down the main diagonal, and f_i is the element in each place down the i-th diagonal above the main diagonal, with zeros elsewhere.

We shall denote by $\{f_0, f_1, \cdots f_{i-1}\}_i^j$ the matrix of i rows and j columns, where f_0 is the element in each place down the counter diagonal and f_i is the element in each place down the ith sub-counter diagonal, with zeros elsewhere. The counter diagonal in a rectangular matrix will always be taken as that diagonal which has its first element in the upper right hand corner of the matrix. If the matrices are square, we shall omit the subscript i and the superscript j. For example

$$(f_0, f_1, f_2, f_3) = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 \\ 0 & f_0 & f_1 & f_2 \\ 0 & 0 & f_0 & f_1 \\ 0 & 0 & 0 & f_0 \end{pmatrix} \qquad \{0, f_1, f_2\}_2^3 = \begin{pmatrix} 0 & 0 \\ 0 & f_1 \\ f_1 & f_2 \end{pmatrix}.$$

In the sequel we shall also have to consider matrices of these two types in which every f_i is a matrix of order two. For convenience we shall use the same notation for both.

We shall employ the notation B^* to represent the conjugate-transposed of the matrix B, i. e. $B^* = \bar{B}'$, and in particular if the elements of B are real, $B^* = B'$. We shall say that a matrix A is transformed into a matrix D by the matrix B, if

$$B^*AB = D.$$

(A transformation of this type is congruent if $B^* = B'$, and is conjunctive if $B^* = \bar{B}'$).

Let $$N_s = (p_j, e, 0, \cdots, 0)$$

be a matrix of order s, and let

$$N_q = (p_j, e, 0, \cdots, 0)$$

be a matrix of order q, where $e = 1$, and p_j is a scalar or else $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and p_j is the real matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $b \neq 0$.

LEMMA (1). *If $RN_q = N'_sR$, where R is a matrix of s rows and q columns, then*

$R = \{f_0, f_1, \cdots, f_{s-1}\}$ *if* $s = q$, *and* $R = \{0, 0, \cdots, f_{s-q}, \cdots, f_{s-1}\}_s^q$ *if* $s > q$. *Moreover if* $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *then* f_i *is a matrix of type* $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$.

Proof. The Lemma has been proved for $e = 1$.† We wish to prove it for $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and in doing so we shall use an induction proof.

† C. E. Cullis, *Matrices and Determinoids*, vol. 3, Part I, pp. 456-459.

Let $R = (r_{ij}) \qquad (i = 1, 2, \cdots, s)$ and $(j = 1, 2, \cdots, q)$

where r_{ij} is a matrix of order two. Since $RN_q = N'_s R$, we have on equating the elements in the i-th rows and j-th columns

$$r_{ij}p_j - p'_j r_{ij} = r_{i-1,j} - r_{i,j-1}, \tag{1}$$

where $r_{0,t} = r_{p,0} = 0 \quad (t = 1, 2, \cdots, s)$ and $(p = 1, 2, \cdots, q)$.

Let us assume that r_{ij} is of the type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$ for all values of i and j for which $i + j \leq k$. Under this assumption it follows from (1) that

$$r_{ij}p_j - p'_j r_{ij} \text{ is of type } \begin{pmatrix} d & c \\ c & -d \end{pmatrix}, \text{ if } i + j = k + 1.$$

If we write $r_{ij} = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$, then

$$r_{ij}p_j - p'_j r_{ij} = \begin{pmatrix} (g_3 - g_2)b & (g_1 + g_4)b \\ -(g_1 + g_4)b & (g_3 - g_2)b \end{pmatrix},$$

and, since $b \neq 0$, this matrix is only of type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$, if $g_1 = -g_4$ and $g_3 = g_2$; i. e. if r_{ij} is of type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$. Hence $r_{ij}p_j - p'_j r_{ij} = 0$, and accordingly

$$r_{i,j-1} = r_{i-1,j}, \qquad i + j \leq k. \tag{2}$$

Since $r_{11}p_j - p'_j r_{11} = 0$, r_{11} is of type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$. Hence our assumption is true for $i + j = 2$ and by induction r_{ij} is of type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$ for all admissible values of i and j, and (2) is true for $i + j \leqq s + q$.

In particular, if $i + j \leqq$ the greater of q and s, since

$$r_{1,j-1} = r_{0j} = 0 \quad (j = 1, 2, \cdots, q) \quad \text{and} \quad r_{i-1,1} = r_{i0} = 0 \quad (i = 1, 2, \cdots, s)$$

it follows that $r_{ij} = 0$.

COROLLARY. *If $R = R^*$, then R is a real matrix.*

For, if $R = R^*$, $f_i = f^*_i$ and since f_i is either a scalar or a matrix of type $\begin{pmatrix} d & c \\ c & -d \end{pmatrix}$, f_i is real.

Let $R = (R_{ij}) \quad (i, j = 1, 2, \cdots, t)$, where $R = R^*$ and R_{ij} is a square matrix of type $\{f_0, f_1, \cdots, f_t\}$, and if each f_i is a two rowed square matrix, it is of type $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.

LEMMA (2). *If in the above matrix* $|R_{11}| = 0$ *and* $|R_{ii}| = 0$, *and if* R_{1i} *has a non-zero element in the upper right-hand corner, then there exists a non-singular matrix* Q *such that* $Q^*RQ = G$, *where* G *is partitioned similarly to* R *and* $|G_{11}| \neq 0$.

Proof. Let us assume the contrary. Without loss of generality we may take R_{1i} to be R_{12}, since by interchanging the 2nd and the i-th columns, and 2nd, and i-th rows, we may move R_{1i} into the place of R_{12}. (Transformations of this type are conjunctive).† Let

$$H = \left[\begin{pmatrix} E & 0 \\ E & E \end{pmatrix}, E, \cdots, E\right],$$

where E is the unit matrix whose order is the same as the order of R_{ij}. Then $H^*RH = K$, where $K_{11} = R_{11} + R_{12} + R^*_{12} + R_{22}$.

Now K_{11} is of type $\{f_0, f_1, \cdots, f_t\}$, and the element in the upper right-hand corner of K_{11} is $r_1 + r_2 + r^*_2 + r_3$ where r_1, r_2, r^*_2 and r_3 are the elements in the upper right-hand corners of R_{11}, R_{12}, R^*_{12}, and R_{22}, respectively. By the corollary to Lemma (1), R_{11} and R_{22} are real matrices, hence if $|R_{11}| = |R_{22}| = 0$, then $r_1 = r_3 = 0$. Since by our assumption $|K_{11}| = 0$, then $|r_2 + r^*_2| = 0$, and as $r_2 + r^*_2$ is real, $r_2 + r^*_2 = 0$. In a similar manner let

$$P = \left[\begin{pmatrix} E & 0 \\ iE & E \end{pmatrix}, E, \cdots, E\right].$$

Then $P^*RP = L$, where $L_{11} = R_{11} + iR_{12} + iR^*_{12} + R_{22}$.

Proceeding in exactly the same manner as above, we have $r_2 - r^*_2 = 0$. Hence $r_2 = 0$, contrary to our hypothesis.

LEMMA (3). *Let* $R = (R_{ij})$ $(i, j = 1, 2, \cdots, t)$, *where* $R = R^*$ *and* R_{ij} *is a matrix with* r_i *rows and* r_j *columns. If* $|R_{11}| \neq 0$ *there exists a matrix* W, *such that*

$$W^*RW = \begin{pmatrix} R_{11} & 0 & 0 & \cdots & 0 \\ 0 & Z_{22} & Z_{23} & \cdots & Z_{2t} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & Z_{t2} & Z_{t3} & \cdots & Z_{tt} \end{pmatrix}.$$

Proof. Let

$$Y_i = \begin{pmatrix} E_1 & 0 & \cdots & -R_{11}^{-1}R_{1i} & \cdots & 0 \\ 0 & E_2 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & E_i & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \cdots & E_t \end{pmatrix},$$

† Cf. Turnbull and Aitken, *An Introduction to the Theory of Canonical Matrices*, p. 11.

where E_i is the unit matrix of order r_i. Then

$$Y^*_t Y^*_{t-1} \cdots Y^*_1 R Y_1 Y_2 \cdots Y_t = \begin{pmatrix} R_{11} & 0 \\ 0 & F_1 \end{pmatrix},$$

where $F_1 = (Z_{ij})$ $(i, j = 2, 3, \cdots, t)$.

LEMMA (4). *If F is the real matrix $\{f_0, f_1, \cdots, f_t\}$, there exists a real non-singular matrix*

$$S = (e, s_1, \cdots, s_t)$$

such that

$$S^*FS = \{f_0, 0, \cdots, 0\},$$

where s_i is of type $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, if each f_i is a two rowed square matrix of the type $\begin{pmatrix} c & d \\ d & -c \end{pmatrix}$.

Proof. Now

$$F = \sum_{i=0}^{t} f_i T_{i+1},$$

where

$$T_r = \sum_{a=r}^{k} e_{a,k+r-a}, \qquad (r = 1, 2, \cdots, t),$$

and e_{ij} is the matrix all of whose elements are zero except that element in the i-th row and j-th column which is e. Likewise

$$S = I_0 + \sum_{i=1}^{t} I_i s_i,$$

where

$$I_j = \sum_{a=1}^{t} e_{a,a+j}, \qquad (j = 0, 1, 2, \cdots, t).$$

Thus it follows that

$$T_r I_j = I'_j T_r = T_{r+j}.$$

Let us assume that there exists a matrix S_{r-1} of the same type as S, such that

$$S^*_{r-1} F S_{r-1} = G = f_0 T_1 + g_{r+1} T_{r+1} + \sum_{a=r+2}^{t+1} g_a T_a,$$

where each g_i is of the same type as f_i. Now

$$(I_0 + s^*_r I'_r) G (I_0 + s_r I_r) = f_0 T_1 + g_{r+1} T_{r+1} + s^*_r f_0 T_{r+1} + f_0 s_r T_{r+1} + \sum_{a=r+2}^{t+1} h_a T_a.$$

Hence on equating the coefficients of T_{r+1} to zero, we deduce that

$$g_{r+1} + s^*_r f_0 + f_0 s_r = 0. \tag{3}$$

But, from the nature of f_i and g_i, it follows that (3) is satisfied if

$$s_r = -\tfrac{1}{2} f_0^{-1} g_{r+1},$$

and that s_r is a matrix of type $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, if f_0 and g_{r+1} are two rowed square matrices. If

$$S_{r-1}(I_0 + s_r I_r) = S_r,$$

where s_r has the value, $s_r = -\tfrac{1}{2} f_0^{-1} g_{r+1}$, then S_r is of the same type as S and

$$S^*_r R S_r = f_0 T_1 + k_{r+2} T_{r+2} + \sum_{a=r+3}^{t+1} k_a T_a,$$

where if f_0 and g_{r+1} are two rowed square matrices then k_i is a matrix of type $\begin{pmatrix} c & d \\ d & -c \end{pmatrix}$. The truth of our Lemma follows immediately by induction, since this formula holds for $r = 1$ $(s_0 = e)$.

2. We consider the non-singular pencil of matrices $A + \lambda B$, in which $|B| \neq 0$, and where $A = A^*$ and $B = B^*$, so that either A or B are both real symmetric matrices or both Hermitian matrices. If $(\lambda + \lambda_s)^\rho$ appears among the elementary divisors of $A + \lambda B$ and if λ_s is complex, then $(\lambda + \bar{\lambda}_s)^\rho$ is also an elementary divisor. Hence we may arrange the elementary divisors of $A + \lambda B$ in such a way that the real elementary divisors are given by

$$(\lambda + \lambda_i)^{\eta_{i1}} (\lambda + \lambda_i)^{\eta_{i2}} \cdots (\lambda + \lambda_i)^{\eta_{ik_i}}, \qquad (i = 1, 2, \cdots, t)$$

and the complex elementary divisors by

$$(\lambda + \lambda_i)^{\eta_{i1}} (\lambda + \bar{\lambda}_i)^{\eta_{i1}} \cdots (\lambda + \lambda_i)^{\eta_{ik_i}} (\lambda + \bar{\lambda}_i)^{\eta_{ik_i}}, \quad (i = t+1, \cdots, s),$$

where $k_1 + k_2 + \cdots + k_t + 2k_{t+1} + \cdots + 2k_s = n$. As is customary we assume that $\eta_{jp} \geq \eta_{jp+1}$ for all values of j and p.

Let $$M = [M_1, M_2, \cdots, M_s]$$

be a real canonical form † for the matrix AB^{-1}, where

$$M_j = [N_{j1}, N_{j2}, \cdots, N_{jk_j}]$$

and $$N_{ji} = (p_j, e, 0, 0, \cdots, 0)$$

† Although this differs slightly from the canonical forms usually listed, it is obviously a real canonical form, since the elementary divisors of $M + \lambda E$ are the same as those of $AB^{-1} + \lambda E$. Cf. Turnbull and Aitken, *An Introduction to the Theory of Canonical Matrices*, p. 72.

is a matrix of order η_{ji}. Hence $e = 1$ and $p_j = \lambda_j$ if λ_j is real, and $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $p_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$ if $\lambda_j = a_j + ib_j$. Thus there exists a non-singular matrix K such that

$$K^{-1}(AB^{-1} + \lambda E)K = M + \lambda E, \tag{4}$$

where, if A and B are real, K is also real. From (4) it follows that

$$K^{-1}(A + \lambda B)B^{-1}K = M + \lambda E$$

and, accordingly, that

$$K^*(B^{-1})^*(A + \lambda B)B^{-1}K = K^*(B^{-1})^*K(M + \lambda E),$$

or more concisely

$$P^*(A + \lambda B)P = F(M + \lambda E), \tag{5}$$

where $P = B^{-1}K$ and $F = K^*(B^{-1})^*K$. Since $B = B^*$, it follows that $F = F^*$, and from (5) that $FM = M^*F$. If $F = (F_{ij})$ is a partition of F similar to that of M, since $FM = M^*F$ we have the result

$$F_{ij}M_j = M^*{}_iF_{ij}, \qquad (i, j = 1, 2, \cdots, s).$$

Since no latent root of $M^*{}_i$ is the same as a latent root of M_j,

$$F_{ij} = 0, \qquad i \neq j.\dagger$$

Thus the matrix F reduces to the diagonal block matrix $[F_{11}, \cdots, F_{ii}, \cdots, F_{ss}]$, where $F^*{}_{ii} = F_{ii}$ and $F_{ii}M_i = M^*{}_iF_{ii}$.

We now consider the matrix equation $F_{ii}M_i = M^*{}_iF_{ii}$, and to simplify our notation, we write $N_j = N_{ij}$ and $k = k_i$. Hence

$$F_{ii}M_i = F_{ii}[N_1, N_2, \cdots, N_k]. \tag{6}$$

If $F_{ii} = (D_{rj})$ is a partition of F_{ii} similar to that of M_i we deduce from (6) that

$$D_{rj}N_j = N^*{}_rD_{rj} = N'{}_rD_{rj}.$$

Hence by Lemma (1) D_{rj} is a matrix of type $\{f_0, f_1, \cdots, f_{r-1}\}^j$, where f_i is of the type $\begin{pmatrix} a & c \\ c & -a \end{pmatrix}$ if every f_i is a two rowed square matrix. In particular D_{jj} is real and of type $\{f_0, f_1, \cdots, f_r\}$.

If the order of N_1 is greater than the order of N_2, D_{11} is non-singular, since F_{ii} is non-singular, and by Lemma (1) D_{1t} $(t = 1, 2, \cdots, k)$ has a

† C. E. Cullis, *loc. cit.*

zero element in the upper right-hand corner. If the order of N_1 is equal to the order of N_2, i. e. if $k_1 = k_2$, let

$$k_1 = k_2 \cdots = k_q > k_{q+1}.$$

If $|D_{11}| = 0$, there is at least one matrix D_{1j} $(1 < j \leq q)$ which has a non-zero element in the upper right-hand corner. If $|D_{11}| = 0$ and $|D_{jj}| \neq 0$ $(j = 2, 3, \cdots, q)$, then by a simple interchange of rows and columns we may move D_{jj} into the place of D_{11}. Again if $|D_{11}| = 0$, and $|D_{jj}| = 0$, then by Lemma (2) there exists a matrix X_i permutable with M_i such that

$$X^*{}_i F_{ii} X_i = T_i,$$

where $|T_{11}| \neq 0$. Hence without loss of generality we may assume that $|D_{11}| \neq 0$. Now by Lemma (3) there exists a matrix L_i permutable with M_i such that

$$L^*{}_i F_{ii} L_i = \begin{pmatrix} D_{11} & 0 \\ 0 & D_1 \end{pmatrix}$$

where $D_1 = (H_{ij})$ $(i, j = 2, 3, \cdots, k)$. Since $\begin{pmatrix} D_{11} & 0 \\ 0 & D_1 \end{pmatrix} M_i$ is Hermitian

$$H_{ij} N_j = N^*{}_i H_{ij} \qquad (i, j = 2, 3, \cdots, k).$$

Thus, we can treat D_1 in the same manner as F_{ii} and in k steps deduce the existence of a non-singular matrix G_i such that

$$(7) \qquad G^*{}_i F_{ii} M_i G_i = G^*{}_i F_{ii} G_i M_i = [K_1, K_2, \cdots, K_k] M_i.$$

Thus we have found a non-singular matrix G_i permutable with M_i such that (7) is true, where $K^*{}_j = K_j$, and $K_j N_j = N^*{}_j K_j$.

Hence by the corollary to Lemma (1), K_j is a real matrix of type $\{f_0, f_1, \cdots, f_r\}$. Therefore by Lemma (4) there exists a matrix S_{ij} permutable with N_j such that

$$(8) \qquad S^*{}_{ij} K_j N_j S_{ij} = S^*{}_{ij} K_j S_{ij} N_j = (f_0 T_1) N_j.$$

We have thus shown that there exists a matrix Q permutable with M such that

$$Q^* P^* (A + \lambda B) P Q = Q^* F (M + \lambda E) Q = Q^* F Q (M + \lambda E)$$

where $Q^* F Q = J$ and J is the diagonal block matrix

$$J = [J_1, J_2, \cdots, J_s],$$

and J_p is a matrix of the diagonal block type

$$J_p = [J_{p1}, J_{p2}, \cdots, J_{pk_p}]$$

and

$$J_{pj} = f_{pj} T_1 \tag{9}$$

is a matrix of η_{pj} rows and columns.

It might be well to note here that Q is a product of matrices of the type

$$GS, \quad \text{where} \quad G = [G_1, G_2, \cdots, G_s] \quad \text{as in (7)}$$

and $S = [S_1, S_2, \cdots, S_s]$, where $S_i = [S_{i1}, S_{i2}, \cdots, S_{ik}]$ as in (8).

If λ_i is real, then in (9) f_{ij} is a real number different from zero. On transforming the matrix $(f_{ij}T_1)N_j$ by the matrix $V_{ij} = (|f_{ij}|^{-\frac{1}{2}} I_0)$ we derive

$$V^*_{ij}(f_{ij}T_1)N_jV_{ij} = V^*_{ij}(f_{ij}T_1)V_{ij}N_j = (\epsilon_{ij}T_1)N_j$$

where $\epsilon_{ij} = \pm 1$. If λ_i is complex, f_{ij} is of the type $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. If $a \neq 0$, on transforming the matrix $(f_{ij}T_1)N_j$ by the matrix $U_{ij} = \begin{pmatrix} r & -1 \\ 1 & r \end{pmatrix} I_0$, where $r = -[b \pm (b^2 + a^2)^{\frac{1}{2}}]/a$,

$$U^*_{ij}(f_{ij}T_1)N_jU_{ij} = U^*_{ij}(f_{ij}T_1)U_{ij}N_j = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} T_1N_j, \tag{10}$$

where k is a positive or negative real number. If $a = 0$, we see that the matrix $(f_{ij}T_1)N_j$ is already in the form (10) with $k = b$. If k is negative we transform $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} T_1N_j$ by the matrix $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} I_0$ and this transformation gives

$$W^*_{ij} \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} T_1N_jW_{ij} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} T_1N_j, \qquad k > 0.$$

If we now transform $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} T_1N_j$ by the matrix $X_{ij} = \begin{pmatrix} k^{-\frac{1}{2}} & 0 \\ 0 & k^{-\frac{1}{2}} \end{pmatrix} I_0$, we have

$$X^*_{ij} \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} N_jX_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T_1N_j.$$

Thus we have found a matrix Y permutable with M, such that

$$Y^*Q^*P^*(A + \lambda B)PQY = Y^*J(M + \lambda E)Y = Y^*JY(M + \lambda E),$$

where $Y^*JY = Z$, and Z is the diagonal block matrix

$$Z = [Z_1, Z_2, \cdots, Z_s] \quad \text{with} \quad Z_i = [Z_{i1}, Z_{i2}, \cdots, Z_{ik_i}]$$

and $Z_{ij} = \epsilon_{ij}T_1$, where $\epsilon_{ij} = \pm 1$ if λ_i is real and $\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if λ_i is complex. Here

$$Y = [Y_1, Y_2, \cdots, Y_s] \quad \text{where} \quad Y_i = [V_{i1}, V_{i2}, \cdots, V_{ik_i}]$$

if $i \leqq t$, and $Y_i = [Y_{i1}, Y_{i2}, \cdots, Y_{ik_i}]$ if $i > t$. If $a \neq 0$

$Y_{ij} = U_{ij}X_{ij}$ $(j = l_1, l_2, \cdots, l_r)$, and $Y_{ij} = U_{ij}W_{ij}X_{ij}$ $(j = l_s, l_{s+1}, \cdots, l_v)$.

If $a = 0$,

$$Y_{ij} = W_{ij}X_{ij} \ (j = t_1, t_2, \cdots, t_s), \quad \text{and} \quad Y_{ij} = X_{ij} \ (j = t_{s+1}, \cdots, t_p),$$

where $l_1, \cdots, l_r, l_s, \cdots, l_v, t_1, \cdots, t_s, \cdots, t_p$ is some permutation of $1, 2, \cdots, k_i$.

Summing up our results we may state: Given a non-singular pencil of matrices $A + \lambda B$, having the properties stated in paragraph 2, we can find a non-singular matrix H, where $HM = MH$ with elements independent of λ such that

$$(11) \quad H^*P^*(A + B)PH = Z(M + \lambda E) = [Z_1, Z_2, \cdots, Z_s](M + \lambda E),$$

where $Z_i = [Z_{i1}, Z_{i2}, \cdots, Z_{ik_i}]$ and $Z_{ij} = \epsilon_{ij}T_1$, where $\epsilon_{ij} = \pm 1$ if λ_i is real and $\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if λ_i is complex.

3. However, it is quite possible that $A + \lambda B$ be equivalent to two different normal forms (11). Let $V(M + \lambda E)$ and $Z(M + \lambda E)$ be two normal forms both equivalent to $A + \lambda B$, where $V = [V_1, V_2, \cdots, V_s]$ and $V_i = [V_{i1}, V_{i2}, \cdots, V_{ik_i}]$, where $V_{ij} = \delta_{ij}T_1$, where $\delta_{ij} = \pm 1$ if λ_i is real and $\delta_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if λ_i is complex. Then there exists a non-singular matrix R such that $R^*VMR = ZM$, and, since in obtaining the normal forms only transformations permutable with M were employed, it follows that

$$(12) \quad RM = MR.$$

Hence $$R^*VR = Z.$$

If the matrices R, V, and F are similarly partitioned, since $RM = MR$, R is a diagonal block matrix and

$$R^*_{ii}V_iR_{ii} = Z_i.$$

Since $V_i = Z_i$ for the complex roots, we need only consider the cases where $i \leq t$. If for simplicity we write R, V, Z and M for R_{ii}, V_i, Z_i and M_i respectively, M is the diagonal block matrix

$$M = [N_1, N_2, \cdots, N_k],$$

where $N_{ij} = N_j$ and N_j is a square matrix of order η_j. Let $R = (S_{ij})$ be a similar partition of R. Then, on equating the elements in the i-th row and j-th column of $R^*VR = Z$, we find that

$$\sum_{j=1}^{k} S^*_{ji}V_jS_{ji} = Z_i.$$

It follows from (12) that †

$$S_{ij} = (r_{ij}, x, y, \cdots, w),$$

where r_{ij} is zero if $\eta_i > \eta_j$. Thus the element in the upper right-hand corner of $S^*{}_{ji}V_jS_{ji}$ is zero unless $\eta_i = \eta_j$. Hence the element in the upper right-hand corner of $\sum_{j=1}^{k} S^*{}_{ji}V_jS_{ji}$ is

$$\sum \bar{r}_{ji}\delta_j r_{ji}, \tag{13}$$

where j ranges over all values for which $\eta_i = \eta_j$. Let

$$\eta_{i-1} > \eta_i = \eta_{i+1} = \cdots = \eta_f > \eta_{f+1}.$$

It follows from (13) that

$$\sum_{t=i}^{f} \bar{r}_{tp}\delta_t r_{tp} = \epsilon_p \quad \text{and} \quad \sum_{t=i}^{f} \bar{r}_{tp}\delta_t r_{tq} = 0 \qquad (p, q = i, i+1, \cdots, f). \tag{14}$$

The two sets of equations in (14) can be written in the following manner

$$\begin{pmatrix} \bar{r}_{ii} & \bar{r}_{i+1,i} & \cdots & \bar{r}_{fi} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \bar{r}_{if} & \bar{r}_{i+1,f} & \cdots & \bar{r}_{ff} \end{pmatrix} \begin{pmatrix} \delta_i & & \cdots & 0 \\ 0 & \delta_{i+1} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \delta_f \end{pmatrix} \begin{pmatrix} r_{ii} & r_{i,i+1} & \cdots & r_{if} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r_{fi} & r_{f,i+1} & \cdots & r_{ff} \end{pmatrix} = \begin{pmatrix} \epsilon_i & & \cdots & 0 \\ 0 & \epsilon_{i+1} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \epsilon_f \end{pmatrix}.$$

Thus the two matrices $[\delta_i, \delta_{i+1}, \cdots, \delta_f]$ and $[\epsilon_i, \epsilon_{i+1}, \cdots, \epsilon_f]$ are equivalent under conjunctive transformation, and therefore

$$\delta_i + \delta_{i+1} + \cdots + \delta_f = \epsilon_i + \epsilon_{i+1} + \cdots + \epsilon_f.$$

Accordingly, if $(\lambda + \lambda_i)^{\eta_{ij}}$ be a real elementary divisor of $A + \lambda B$, where $|B| \neq 0$, in the normal form there is associated with this elementary divisor an $\epsilon_{ij} = \pm 1$. If $(\lambda + \lambda_i)^{\eta_{ij}}$ is repeated exactly p times among the elementary divisors of $A + \lambda B$, we denote the ϵ_{ij} associated with these p elementary divisors by ${}^1\epsilon_{ij}, {}^2\epsilon_{ij}, \cdots, {}^p\epsilon_{ij}$ and write

$$\sigma_{ij} = \sum_{\alpha=1}^{p} {}^\alpha\epsilon_{ij}. \tag{15}$$

Now if s of the ${}^\alpha\epsilon_{ij}$ have the value $+1$ and $p - s$ have the value -1, in the normal forms there are $p!/s!\,(p-s)!$ possible arrangements of the ${}^\alpha\epsilon_{ij}$, all of which are equivalent by our last result. We choose for *the normal form* that one in which the first s are positive. Thus two non-singular pencils of matrices $A + \lambda B$ and $C + \lambda D$, both having the properties of paragraph 2, are equivalent under congruent or conjunctive transformation if and only if they have the same normal form, i. e. if

† C. E. Cullis, *loc. cit.*

(1) they have the same elementary divisors,

(2) the integers σ_{ij} defined by (15) are the same for both pencils.†

If in the pencil of matrices $A + \lambda B$ both A and B are singular, but $|A + \lambda B| \not\equiv 0$, we consider the non-singular pencil $\lambda_1 A + \lambda_2 B$. Let

$$\lambda_1 = g\lambda - g' \quad \text{and} \quad \lambda_2 = h\lambda - h',$$

where g, h, g' and h' are so chosen that

$$|gA + hB| \not\equiv 0 \quad \text{and} \quad gh' - g'h = 1.$$

Hence $$\lambda_1 A + \lambda_2 B = \lambda(gA + hB) - (g'A + h'B) = A_1 + \lambda B_1.$$

In a similar manner

$$\lambda_1 C + \lambda_2 D = \lambda(gC + hD) - (g'C + h'D) = C_1 + \lambda D_1.$$

Now $\lambda_1 A + \lambda_2 B$ and $\lambda_1 C + \lambda_2 D$ are equivalent if and only if $A_1 + \lambda B_1$ and $C_1 + \lambda D_1$ are equivalent. Thus we may state:

A necessary and sufficient condition that two non-singular pencils of matrices $\lambda_1 A + \lambda_2 B$ and $\lambda_1 C + \lambda_2 D$ (where $A = A^*$, $B = B^*$, $C = C^*$ and $D = D^*$ so that either both A and B, also both C and D, are real symmetric matrices or Hermitian matrices) be equivalent under congruent or conjunctive transformations is that

(1) *they have the same elementary divisors,*

(2) *the integers σ_{ij} be the same for both pencils.* Here σ_{ij} is defined for the pencil $A_1 + \lambda B_1$ and $C_1 + \lambda D_1$ in a manner analogous to that in (15).

At the time of the completion of this paper, an abstract by K. W. Wegner ‡ appeared as follows:

"A necessary and sufficient condition that two pairs of Hermitian matrices $A + \lambda B$ and $C + \lambda D$, $|B| \not\equiv 0$ and $|D| \not\equiv 0$, be equivalent under conjunctive transformations is that

(α) they have the same elementary divisors,

(β) the indices of $B(B^{-1}A + \lambda E)^m$ and $D(D^{-1}C + \lambda E)^m$ be the same for every real value of λ and every positive integral value of m."

We wish to show here that the results obtained by Wegner and the results obtained in this paper are equivalent. From (4) we have

$$K^{-1}(AB^{-1} + \lambda E)K = M + \lambda E.$$

Hence $$K^{-1}(AB^{-1} + \lambda E)^m K = (M + \lambda E)^m.$$

† In the real case, this result is the same as the result derived by P. Muth, "Ueber reele Aquivalenz von Scharen reeler quadratischer Formen." *Crelle's Journal*, p. 314.

‡ *Bulletin of the American Mathematical Society*, January, 1934.

Thus we can write

$$K^{-1}(AB^{-1}+\lambda E)^m BB^{-1}K=(M+\lambda E)^m$$

and accordingly

$$P^*(AB^{-1}+\lambda E)^m BP=F(M+\lambda E)^m,$$

where P has the same meaning as in (5). We may proceed exactly as we did in the derivation of our normal form and deduce that

$$Q^*P^*(AB^{-1}+\lambda E)^m BPQ=Q^*FQ(M+\lambda E)^m=Z(M+\lambda E)^m.$$

It is obvious that if the two pencils have the same normal form, the results (α) and (β) are true.†

For a fixed value of λ and a fixed value of m, it is apparent that there exists a non-singular matrix K, such that

$$K^*(AB^{-1}+\lambda E)^m BK=H, \quad \text{where} \quad H=[H_1, H_2, \cdots, H_s] \quad \text{and}$$
$$H_i=[H_{i1}, H_{i2}, \cdots, H_{ik_i}] \quad \text{and} \quad H_{ij}=\epsilon_{ij}(\lambda+\lambda_i)^m T_1.$$

If $i>t$, the signature of H_{ij} is obviously zero. If $i\leq t$ and $\lambda+\lambda_i$ is positive, the signature of H_{ij} is independent of the value of m. Let us suppose λ_1 so chosen that if $\lambda=-\lambda_1$, $\lambda_j+\lambda>0$ $(j=2, 3, \cdots, t)$. Let s_i be the signature of $Z_i(M_i-\lambda_1 E)^m$, and s'_i be the signature of $V_i(M_i-\lambda_1 E)^m$ $(i=2, 3, \cdots, t)$. If $m=\eta_{11}$,

$$Z_1(M_1-\lambda_1 E)^m=0,$$

and, since the signature of $Z(M+\lambda E)^m$ and $V(M+\lambda E)^m$ are the same for all real values of λ and every integral value of m, we have

$$s_2+s_2+\cdots+s_t=s'_2+s'_3+\cdots+s'_t.$$

If $m=\eta_{11}-1$, the signature of $Z_1(M_1-\lambda_1 E)^m$ is $\Sigma^a\epsilon_{11}=\sigma_{11}$, and the signature of $V_i(M_1+\lambda_1 E)^m$ is $\Sigma^a\delta_{11}=\sigma'_{11}$. Therefore

$$\Sigma^a\epsilon_{11}+s_2+\cdots s_t=\Sigma^a\delta_{11}+s'_2\cdots+s'_t.$$

Hence $$\Sigma^a\epsilon_{11}=\Sigma^a\delta_{11}, \quad \text{i. e.} \quad \sigma_{11}=\sigma'_{11}.$$

On continuing this process, we easily see that $\sigma_{ij}=\sigma'_{ij}$ for every value of i and j. Thus if (α) and (β) are true, the two pencils have the same normal form.

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† The matrix which we transform here differs from that in (β) only as to the situation of B, hence a change of notation would be necessary to alter this slight difference; but it is obvious that the work carries through in the same manner, so we refer to (β), keeping this difference in mind.

VITA

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